Numerical solution of the (Stokes) viscous boundary layer

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1 Differential equation

The Stokes layer of the velocity satisfies the following differential equation:

$$\frac{\partial \hat{u}}{\partial t} - \frac{\mu}{\rho_0}\nabla_{\mathrm{T}}^2 \hat{u} = \hat{K}(t), \tag{1}$$

where

- t is time,
- ∇_{T}^2 is the Laplacian in *transverse* direction.
- $\hat{u}$ is the *axial* velocity,
- $\hat{K}(t)$ is the forcing function as a function of time, which in acoustics equals minus the gradient of the pressure,
- μ is the dynamic viscosity,
- ρ_0 the density.

2 Harmonic solution

If we assume harmonic motion, we may write

$$\hat{u} = \Re\left(ue^{i\omega t}\right) \quad ; \quad K = \Re\left(\hat{K}e^{i\omega t}\right) \quad \text{etc} \tag{2}$$

where

- $\Re$ is the operator taking the real part of its argument,
- $i = \sqrt{-1}$,
- ω is the frequency in rad/s.

For flow close to a plate and unbounded in the $+y$-direction, the solution for u yields:

$$u = \frac{1}{i\omega\rho_0}K\left(1 - \exp\left(-\left(1+i\right)y/\delta_\nu\right)\right), \tag{3}$$

where

- $\delta_\nu = \sqrt{\frac{2\mu}{\rho_0\omega}}$

For oscillating flow between two parallel plates, separated at distances $2y_0$, and $y = 0$ at the center between the two plates:

$$u = \frac{1}{i\omega\rho_0}\frac{1-h_\nu}{1-f_\nu}K, \tag{4}$$

where

- $h_\nu = \frac{\cosh((1+i)y/\delta_\nu)}{\cosh((1+i)y_0/\delta_\nu)}$,
- $f_\nu = \frac{\tanh((1+i)y_0/\delta_\nu)}{(1+i)y_0/\delta_\nu}$.

3 Forward Euler time, central in space

For $\nabla_{\mathrm{T}}^2 \equiv \frac{\partial^2}{\partial y^2}$, the forward Euler time, central in space (uniform grid) formulation of Eq. 1 is

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{\mu}{\rho_0}\frac{u_{i+1}^n - 2u_i^n - u_{i-1}^n}{\Delta y^2} = K^n, \tag{5}$$

where upper index n denotes a discrete time instance and lower index i denotes a discrete position index. This is an explicit form for the velocity at the next time index $n+1$. Solving for u_i^{n+1} yields:

$$u_i^{n+1} = u_i^n + \Delta t K^n + \frac{\mu\Delta t}{\rho_0\Delta y^2}\left(u_{i+1}^n - 2u_i^n - u_{i-1}^n\right) \tag{6}$$